

Quantitative uniqueness for second order elliptic operators with strongly singular coefficients

Ching-Lung Lin^{*} Gen Nakamura[†] Jenn-Nan Wang[‡]

Abstract

In this paper we study the local behavior of a solution to second order elliptic operators with sharp singular coefficients in lower order terms. One of the main results is the bound on the vanishing order of the solution, which is a quantitative estimate of the strong unique continuation property. Our proof relies on Carleman estimates with carefully chosen phases. A key strategy in the proof is to derive doubling inequalities via three-sphere inequalities. Our method can also be applied to certain elliptic systems with similar singular coefficients.

1 Introduction

Assume that Ω is a connected open set containing 0 in \mathbb{R}^n for $n \geq 2$. Let $P(x, D) = \sum_{j,k} a_{jk}(x) D_j D_k$ be an elliptic differential operator in Ω such that $a_{jk}(0)$ is a real symmetric matrix and $a_{jk}(x)$ is Lipschitz continuous in Ω , where $D_j = \partial/\partial x_j$, $j = 1, \dots, n$. Note that $a_{jk}(x)$ could be complex valued

^{*}Department of Mathematics, National Chung Cheng University, Chia-Yi 62117, Taiwan.

Partially supported by the National Science Council of Taiwan.

[†]Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan.
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[‡]Department of Mathematics, Taida Institute of Mathematical Sciences, NCTS (Taipei), National Taiwan University, Taipei 106, Taiwan.
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at $x \neq 0$. In this paper we consider the following second order differential inequality:

$$|P(x, D)u| \leq \frac{C_1}{|x|^2}|u| + \frac{C_2}{|x|}|\nabla u| \quad \text{in } \Omega, \quad (1.1)$$

where C_2 is sufficiently small. Before proceeding to the main discussion, we want to point out that restrictions described above are necessary. It is well known that the Lipschitz smoothness requirement on a_{ij} is minimal for the unique continuation to hold [14]. Counterexamples given by Alinhac [2] show that the restriction of $a_{ij}(0)$ being real is necessary for the strong unique continuation. On the other hand, regarding the constant C_2 , the strong unique continuation fails for (1.1) if C_2 is not small, see [3] and [16]. Finally, simple counterexamples also show that the singular coefficients on the right side of (1.1) are sharp for the strong unique continuation. Under the same assumptions, the strong unique continuation property for (1.1) was proved by Regbaoui [15]. But Regbaoui did not give any quantitative estimate on the vanishing order of u satisfying (1.1). This is our main goal in this work. The development of qualitative unique continuation property has a long history. We do not intend to give a summary here. We refer to the paper [10] and references therein for more details.

Concerning about the quantitative estimate of the uniqueness for partial differential operators, we would like to mention several related works. Using the frequency function, Garofala and Lin [5], [6] derived a quantitative version of the strong unique continuation for strongly second order elliptic operators. In [5], they also considered $|x|^{-2}$ potentials but without first order terms. In [6], they studied full lower order terms with certain singular coefficients, but they are not sharp. Also in [11], Kukavica used the frequency function to prove the maximal vanishing order of solutions to the strong second order elliptic operator with essentially bounded potentials. Our method in this paper is different from those in [5], [6], and [11]. Our key tools are Carleman estimates. Besides of the difference in method, the differential operator $P(x, D)$ in (1.1) is only elliptic and the coefficients on the right hand side of (1.1) are strongly singular. None of [5], [6], and [11] dealt with the equation as (1.1).

On the other hand, Donnelly and Fefferman [4] applied Carleman's technique to derive the maximal vanishing order of the eigenfunction with respect to the corresponding eigenvalue on a compact smooth Riemannian manifold. Also, in [12], Lin applied the Carleman estimate proved by Jerison and Kenig

[9] to derive a quantitative estimate of the strong unique continuation property for the Schrödinger equation with $L_{loc}^{n/2}$ potential. However, the methods in [4] and [12] can not be applied to (1.1) with strongly singular coefficients. The difficulty lies in the fact that all Carleman estimates used to treat the strong unique continuation contain only polynomial weights, which are not "singular" enough to handle sharp singular coefficients in the lower derivatives. In this work, we overcome this difficulty by deriving three-sphere inequalities using slightly singular than polynomial weights. Then we proceed to derive doubling inequalities and the bound on the vanishing order of the solution to (1.1) by applying three-sphere inequalities recursively.

In this paper, for brevity, we only consider the scalar second order elliptic operator. But our method can also be applied to the case where $P(x, D)$ is an elliptic system as

$$P(x, D) = \text{diag}(P_1(x, D), \dots, P_\ell(x, D)),$$

where $P_j(x, D)$, $j = 1, \dots, \ell$, are second order elliptic operators with Lipschitz coefficients and satisfy that $P_j(0, D) = \dots = P_\ell(0, D)$ with real symmetric coefficients. All methods mentioned above do not seem to work in this general case. Finally, we would like to mention that quantitative estimates of the strong unique continuation are useful in studying the nodal sets of eigenfunctions [4], or solutions of second order elliptic equations [7], [13], or the inverse problem [1]. The main results of the paper are summarized as follows. Assume that $B_{R_0} \subset \Omega$.

Theorem 1.1 *There exists a positive number $R_1 < 1$ such that if $0 < r_1 < r_2 < r_3 \leq R_0$ and $r_1/r_3 < r_2/r_3 < R_1$, then*

$$\int_{|x| < r_2} |u|^2 dx \leq C \left(\int_{|x| < r_1} |u|^2 dx \right)^\tau \left(\int_{|x| < r_3} |u|^2 dx \right)^{1-\tau} \quad (1.2)$$

for $u \in H^1(B_{R_0})$ satisfying (1.1) in B_{R_0} , where C and $0 < \tau < 1$ depend on r_1/r_3 , r_2/r_3 and $P(x, D)$.

Remark 1.1 *From the proof, it suffices to take $R_1 \leq 1/4$. Moreover, the constants C and τ can be explicitly written as $C = \max\{C_0(r_2/r_1)^n, \exp(B\beta_0)\}$ and $\tau = B/(A+B)$, where $C_0 > 1$ and β_0 are constants depending on $P(x, D)$ and*

$$\begin{aligned} A &= A(r_1/r_3, r_2/r_3) = (\log(r_1/r_3) - 1)^2 - (\log(r_2/r_3))^2, \\ B &= B(r_2/r_3) = -1 - 2\log(r_2/r_3). \end{aligned}$$

The explicit forms of these constants are important in the proof of Theorem 1.2.

Theorem 1.2 *There exists a constant C depending on $P(x, D)$ such that if $u \in H_{loc}^1(\Omega)$ is a nonzero solution to (1.1) with $C_2 < C$, then we can find a constant R_2 depending on $P(x, D)$ and a constant m_1 depending on $P(x, D)$ and $\|u\|_{L^2(|x| < R_2^2)} / \|u\|_{L^2(|x| < R_2^4)}$ satisfying*

$$\limsup_{R \rightarrow 0} \frac{1}{R^{m_1}} \int_{|x| < R} |u|^2 dx > 0. \quad (1.3)$$

In view of the standard unique continuation property for (1.1) in a connected domain containing the origin, if u vanishes in a neighborhood of the origin then it vanishes identically in Ω . Theorem 1.2 provides an upper bound on the vanishing order of a nontrivial solution to (1.1). The following doubling inequality is another quantitative estimate of the strong unique continuation for (1.1).

Theorem 1.3 *Let $u \in H_{loc}^1(\Omega)$ be a nonzero solution to (1.1). Then there exist positive constants R_3 depending on $P(x, D)$, and C_3 depending on $P(x, D)$, m_1 such that if $0 < r \leq R_3$, then*

$$\int_{|x| \leq 2r} |u|^2 dx \leq C_3 \int_{|x| \leq r} |u|^2 dx, \quad (1.4)$$

where m_1 is the constant obtained in Theorem 1.2.

The rest of the paper is devoted to the proofs of Theorem 1.1-1.3.

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. To begin, we recall a Carleman estimate with weight $\varphi_\beta = \varphi_\beta(|x|) = \exp(\frac{\beta}{2}(\log|x|)^2)$ derived in [15].

Lemma 2.1 [15, Theorem 1.2] *For any $\beta > 0$ large enough. Let S be a small neighborhood of 0 and $u : S \setminus \{0\} \subset \Omega \rightarrow \mathbb{R}$, $u \in C_0^\infty(S \setminus \{0\})$. Then we have*

$$\begin{aligned} & \beta^3 \int \varphi_\beta^2 |x|^{-n} |u|^2 dx + \beta \int \varphi_\beta^2 |x|^{-n+2} |\nabla u|^2 dx \\ & \leq \tilde{C}_0 \int \varphi_\beta^2 |x|^{-n+4} |P(x, D)u|^2 dx, \end{aligned} \quad (2.1)$$

for some positive constant \tilde{C}_0 depending only on $P(x, D)$.

Remark 2.1 *The estimate (2.1) in Lemma 2.1 remains valid if we assume $u \in H^2(S \setminus \{0\})$ with compact support. This can be easily obtained by cutting off u for small $|x|$ and regularizing.*

We now proceed to the main part of the proof. Using regularization, Friedrich's lemma, and ellipticity of $P(x, D)$, we can see that if $u \in H_{loc}^1(\Omega)$ satisfies (1.1) then $u \in H_{loc}^2(\Omega \setminus \{0\})$. To begin, we first consider the case where $0 < r_1 < r_2 < R < 1$ and $B_R \subset \Omega$. The constant R will be determined later. To use the Carleman estimate (2.1), we need to cut-off u . So let $\xi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \leq \xi(x) \leq 1$ and

$$\xi(x) = \begin{cases} 0, & |x| \leq r_1/e, \\ 1, & r_1/2 < |x| < er_2, \\ 0, & |x| \geq 3r_2. \end{cases}$$

Here $e = \exp(1)$. It is easy to see that for all multiindex α

$$\begin{cases} |D^\alpha \xi| = O(r_1^{-|\alpha|}) \text{ for all } r_1/e \leq |x| \leq r_1/2 \\ |D^\alpha \xi| = O(r_2^{-|\alpha|}) \text{ for all } er_2 \leq |x| \leq 3r_2. \end{cases} \quad (2.2)$$

On the other hand, repeating the proof of Corollary 17.1.4 in [8], we can show that

$$\int_{a_1 r < |x| < a_2 r} ||x|^{|\alpha|} D^\alpha u|^2 dx \leq C' \int_{a_3 r < |x| < a_4 r} |u|^2 dx, \quad |\alpha| \leq 2, \quad (2.3)$$

for all $0 < a_3 < a_1 < a_2 < a_4$ such that $B_{a_4 r} \subset \Omega$, where the constant C' is independent of r .

Noting that the commutator $[P(x, D), \xi]$ is a first order differential operator. Applying (2.1) to ξu and using (1.1), (2.2), (2.3) implies

$$\begin{aligned} & \beta^3 \int_{r_1/2 < |x| < er_2} \varphi_\beta^2 |x|^{-n} |u|^2 dx + \beta \int_{r_1/2 < |x| < er_2} \varphi_\beta^2 |x|^{-n+2} |\nabla u|^2 dx \\ & \leq \beta^3 \int \varphi_\beta^2 |x|^{-n} |\xi u|^2 dx + \beta \int \varphi_\beta^2 |x|^{-n+2} |\nabla(\xi u)|^2 dx \\ & \leq \tilde{C}_0 \int \varphi_\beta^2 |x|^{-n+4} |P(x, D)(\xi u)|^2 dx \end{aligned}$$

$$\begin{aligned}
&\leq \tilde{C}_0 \int \varphi_\beta^2 (C_1^2 |x|^{-n} |\xi u|^2 + C_2^2 |x|^{-n+2} |\xi \nabla u|^2) dx \\
&\quad + \tilde{C}_0 \int \varphi_\beta^2 |x|^{-n+4} |P(x, D), \xi u|^2 dx \\
&\leq \tilde{C}_1 \left\{ \int_{r_1/2 < |x| < er_2} \varphi_\beta^2 |x|^{-n} |u|^2 dx + \int_{r_1/2 < |x| < er_2} \varphi_\beta^2 |x|^{-n+2} |\nabla u|^2 dx \right. \\
&\quad + \int_{r_1/e < |x| < r_1/2} \varphi_\beta^2 |x|^{-n} |u|^2 dx + \int_{r_1/e < |x| < r_1/2} \varphi_\beta^2 |x|^{-n+2} |\nabla u|^2 dx \\
&\quad \left. + \int_{er_2 < |x| < 3r_2} \varphi_\beta^2 |x|^{-n} |u|^2 dx + \int_{er_2 < |x| < 3r_2} \varphi_\beta^2 |x|^{-n+2} |\nabla u|^2 dx \right\} \\
&\leq \tilde{C}_2 \left\{ \int_{r_1/2 < |x| < er_2} \varphi_\beta^2 |x|^{-n} |u|^2 dx + \int_{r_1/2 < |x| < er_2} \varphi_\beta^2 |x|^{-n+2} |\nabla u|^2 dx \right. \\
&\quad + r_1^{-n} \varphi_\beta^2(r_1/e) \int_{r_1/e < |x| < r_1/2} (|u|^2 + |x|^2 |\nabla u|^2) dx \\
&\quad \left. + r_2^{-n} \varphi_\beta^2(er_2) \int_{er_2 < |x| < 3r_2} (|u|^2 + |x|^2 |\nabla u|^2) dx \right\} \\
&\leq \tilde{C}_3 \left\{ \int_{r_1/2 < |x| < er_2} \varphi_\beta^2 |x|^{-n} |u|^2 dx + \int_{r_1/2 < |x| < er_2} \varphi_\beta^2 |x|^{-n+2} |\nabla u|^2 dx \right. \\
&\quad \left. + r_1^{-n} \varphi_\beta^2(r_1/e) \int_{r_1/4 < |x| < r_1} |u|^2 dx + r_2^{-n} \varphi_\beta^2(er_2) \int_{2r_2 < |x| < 4r_2} |u|^2 dx \right\}, \tag{2.4}
\end{aligned}$$

where \tilde{C}_1 , \tilde{C}_2 , and \tilde{C}_3 are independent of r_1 and r_2 . Now letting $\beta_0 \geq 1$ and $\beta \geq \beta_0 \geq 2\tilde{C}_3$ in (2.4), we immediately get that

$$\begin{aligned}
&\int_{r_1/2 < |x| < er_2} \varphi_\beta^2 |x|^{-n} |u|^2 dx + \int_{r_1/2 < |x| < er_2} \varphi_\beta^2 |x|^{-n+2} |\nabla u|^2 dx \\
&\leq \tilde{C}_4 \left\{ r_1^{-n} \varphi_\beta^2(r_1/e) \int_{r_1/4 < |x| < r_1} |u|^2 dx + r_2^{-n} \varphi_\beta^2(er_2) \int_{2r_2 < |x| < 4r_2} |u|^2 dx \right\}, \tag{2.5}
\end{aligned}$$

where $\tilde{C}_4 = 1/\tilde{C}_3$. It follows easily from (2.5) that

$$r_2^{-n} \varphi_\beta^2(r_2) \int_{r_1/2 < |x| < r_2} |u|^2 dx$$

$$\begin{aligned}
&\leq \int_{r_1/2 < |x| < er_2} \varphi_\beta^2 |x|^{-n} |u|^2 dx \\
&\leq \tilde{C}_4 \left\{ r_1^{-n} \varphi_\beta^2(r_1/e) \int_{r_1/4 < |x| < r_1} |u|^2 dx + r_2^{-n} \varphi_\beta^2(er_2) \int_{2r_2 < |x| < 4r_2} |u|^2 dx \right\}.
\end{aligned} \tag{2.6}$$

Dividing $r_2^{-n} \varphi_\beta^2(r_2)$ on the both sides of (2.6) implies

$$\begin{aligned}
&\int_{r_1/2 < |x| < r_2} |u|^2 dx \\
&\leq \tilde{C}_4 \left\{ (r_2/r_1)^n [\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)] \int_{r_1/4 < |x| < r_1} |u|^2 dx \right. \\
&\quad \left. + [\varphi_\beta^2(er_2)/\varphi_\beta^2(r_2)] \int_{2r_2 < |x| < 4r_2} |u|^2 dx \right\} \\
&\leq \tilde{C}_5 \left\{ (r_2/r_1)^n [\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)] \int_{|x| < r_1} |u|^2 dx \right. \\
&\quad \left. + (r_2/r_1)^n [\varphi_\beta^2(er_2)/\varphi_\beta^2(r_2)] \int_{|x| < 4r_2} |u|^2 dx \right\},
\end{aligned} \tag{2.7}$$

where $\tilde{C}_5 = \max\{\tilde{C}_4, 1\}$. With such choice of \tilde{C}_5 , we see that

$$\tilde{C}_5 (r_2/r_1)^n [\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)] > 1$$

for all $0 < r_1 < r_2$. Adding $\int_{|x| < r_1/2} |u|^2 dx$ to both sides of (2.7) and choosing $r_2 \leq 1/4$, we obtain that

$$\begin{aligned}
&\int_{|x| < r_2} |u|^2 dx \\
&\leq 2\tilde{C}_5 (r_2/r_1)^n [\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)] \int_{|x| < r_1} |u|^2 dx \\
&\quad + 2\tilde{C}_5 (r_2/r_1)^n [\varphi_\beta^2(er_2)/\varphi_\beta^2(r_2)] \int_{|x| < 1} |u|^2 dx.
\end{aligned} \tag{2.8}$$

For simplicity, by denoting

$$\begin{aligned}
A &= \beta^{-1} \log[\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)] = (\log r_1 - 1)^2 - (\log r_2)^2 > 0, \\
B &= -\beta^{-1} \log[\varphi_\beta^2(er_2)/\varphi_\beta^2(r_2)] = -1 - 2 \log r_2 > 0,
\end{aligned}$$

(2.8) becomes

$$\begin{aligned} & \int_{|x|<r_2} |u|^2 dx \\ & \leq 2\tilde{C}_5(r_2/r_1)^n \left\{ \exp(A\beta) \int_{|x|<r_1} |u|^2 dx + \exp(-B\beta) \int_{|x|<1} |u|^2 dx \right\}. \end{aligned} \quad (2.9)$$

To further simplify the terms on the right hand side of (2.9), we consider two cases. If

$$\exp(A\beta_0) \int_{|x|<r_1} |u|^2 dx < \exp(-B\beta_0) \int_{|x|<1} |u|^2 dx,$$

then we can pick a $\beta > \beta_0$ such that

$$\exp(A\beta) \int_{|x|<r_1} |u|^2 dx = \exp(-B\beta) \int_{|x|<1} |u|^2 dx.$$

Using such β , we obtain from (2.9) that

$$\begin{aligned} & \int_{|x|<r_2} |u|^2 dx \\ & \leq 4\tilde{C}_5(r_2/r_1)^n \exp(A\beta) \int_{|x|<r_1} |u|^2 dx \\ & = 4\tilde{C}_5(r_2/r_1)^n \left(\int_{|x|<r_1} |u|^2 dx \right)^{\frac{B}{A+B}} \left(\int_{|x|<1} |u|^2 dx \right)^{\frac{A}{A+B}}. \end{aligned} \quad (2.10)$$

On the other hand, if

$$\exp(-B\beta_0) \int_{|x|<1} |u|^2 dx \leq \exp(A\beta_0) \int_{|x|<r_1} |u|^2 dx,$$

then we have

$$\begin{aligned} & \int_{|x|<r_2} |u|^2 dx \\ & \leq \left(\int_{|x|<1} |u|^2 dx \right)^{\frac{B}{A+B}} \left(\int_{|x|<r_1} |u|^2 dx \right)^{\frac{A}{A+B}} \\ & \leq \exp(B\beta_0) \left(\int_{|x|<r_1} |u|^2 dx \right)^{\frac{B}{A+B}} \left(\int_{|x|<1} |u|^2 dx \right)^{\frac{A}{A+B}}. \end{aligned} \quad (2.11)$$

Putting together (2.10), (2.11), and setting $\tilde{C}_6 = \max\{4\tilde{C}_5(r_2/r_1)^n, \exp(B\beta_0)\}$, we arrive at

$$\int_{|x|<r_2} |u|^2 dx \leq \tilde{C}_6 \left(\int_{|x|<r_1} |u|^2 dx \right)^{\frac{B}{A+B}} \left(\int_{|x|<1} |u|^2 dx \right)^{\frac{A}{A+B}}. \quad (2.12)$$

Now for the general case, we take $R_1 \leq 1/4$ and consider $0 < r_1 < r_2 < r_3$ with $r_1/r_3 < r_2/r_3 \leq 1/4$. By scaling, i.e. defining $\hat{u}(y) := u(r_3 y)$ and $\widehat{a_{ij}}(y) = a_{ij}(r_3 y)$, we derive from (2.12) that

$$\int_{|y|<r_2/r_3} |\hat{u}|^2 dy \leq C \left(\int_{|y|<r_1/r_3} |\hat{u}|^2 dy \right)^\tau \left(\int_{|y|<1} |\hat{u}|^2 dy \right)^{1-\tau}, \quad (2.13)$$

where $\tau = B/(A+B)$ with

$$\begin{aligned} A &= A(r_1/r_3, r_2/r_3) = (\log(r_1/r_3) - 1)^2 - (\log(r_2/r_3))^2, \\ B &= B(r_2/r_3) = -1 - 2\log(r_2/r_3), \end{aligned}$$

and $C = \max\{4\tilde{C}_5(r_2/r_1)^n, \exp(B\beta_0)\}$. We want to remark that \tilde{C}_5 can be chosen independent of the scaling factor r_3 provided $r_3 < 1$. Restoring the variable $x = r_3 y$ in (2.13) gives

$$\int_{|x|<r_2} |u|^2 dx \leq C \left(\int_{|x|<r_1} |u|^2 dx \right)^\tau \left(\int_{|x|<r_3} |u|^2 dx \right)^{1-\tau}.$$

The proof now is complete. \square

3 Proof of Theorem 1.2 and Theorem 1.3

In this section, we prove Theorem 1.2 and Theorem 1.3. Without loss of generality, we assume $P(0, D) = \Delta$ by the change of coordinates. We begin with another Carleman estimate derived in [15, Lemma 2.1]: for any $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and for any $m \in \{j + \frac{1}{2}, j \in \mathbb{N}\}$ we have

$$\sum_{|\alpha| \leq 2} \int m^{2-2|\alpha|} |x|^{-2m+2|\alpha|-n} |D^\alpha u|^2 dx \leq C \int |x|^{-2m+4-n} |\Delta u|^2 dx, \quad (3.1)$$

where C only depends on the dimension n .

Remark 3.1 *Using the cut-off function and regularization, estimate (3.1) remains valid for any fixed m if $u \in H_{loc}^2(\mathbb{R}^n \setminus \{0\})$ with compact support.*

In view of Remark 3.1, we can apply (3.1) to the function χu with $\chi(x) \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. Therefore, we define $\chi(x) \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ such that

$$\chi(x) = \begin{cases} 0 & \text{if } |x| \leq \delta/3, \\ 1 & \text{in } \delta/2 \leq |x| \leq (R_0 + 1)R_0R/4 = r_4R, \\ 0 & \text{if } 2r_4R \leq |x|, \end{cases}$$

where $\delta \leq R_0^2R/4$, $R_0 > 0$ is a small number which will be chosen later and R is sufficiently small satisfying $0 < R \leq R_0$. Here the number R is not yet fixed and is given by $R = (\gamma m)^{-1}$, where $\gamma > 0$ is a large constant which will be chosen later. Using the estimate (3.1) and the equation (1.1), we can derive that

$$\begin{aligned} & \sum_{|\alpha| \leq 2} m^{2-2|\alpha|} \int_{\delta/2 \leq |x| \leq r_4R} |x|^{-2m+2|\alpha|-n} |D^\alpha u|^2 dx \\ & \leq \sum_{|\alpha| \leq 2} m^{2-2|\alpha|} \int |x|^{-2m+2|\alpha|-n} |D^\alpha(\chi u)|^2 dx \\ & \leq C \int |x|^{-2m+4-n} |\Delta(\chi u)|^2 dx \\ & \leq C \int_{\delta/2 \leq |x| \leq r_4R} |x|^{-2m+4-n} |\Delta u|^2 dx + C \int_{|x| > r_4R} |x|^{-2m+4-n} |\Delta(\chi u)|^2 dx \\ & \quad + C \int_{\delta/3 \leq |x| \leq \delta/2} |x|^{-2m+4-n} |\Delta(\chi u)|^2 dx \\ & \leq \hat{C}' \int_{\delta/2 \leq |x| \leq r_4R} |x|^{-2m+4-n} |\Delta u - P(x, D)u|^2 dx \\ & \quad + \hat{C}' \int_{\delta/2 \leq |x| \leq r_4R} |x|^{-2m+4-n} |P(x, D)u|^2 dx \\ & \quad + C \int_{|x| > r_4R} |x|^{-2m+4-n} |\Delta(\chi u)|^2 dx + C \int_{\delta/3 \leq |x| \leq \delta/2} |x|^{-2m+4-n} |\Delta(\chi u)|^2 dx \end{aligned}$$

$$\begin{aligned}
&\leq C' \sum_{|\alpha|=2} r_4^2 R^2 \int_{\delta/2 \leq |x| \leq r_4 R} |x|^{-2m+4-n} |D^\alpha u|^2 dx \\
&\quad + C' C_1^2 \int_{\delta/2 \leq |x| \leq r_4 R} |x|^{-2m-n} |u|^2 dx + C' C_2^2 \sum_{|\alpha|=1} \int_{\delta/2 \leq |x| \leq r_4 R} |x|^{-2m+2-n} |D^\alpha u|^2 dx \\
&\quad + C \int_{|x| > r_4 R} |x|^{-2m+4-n} |\Delta(\chi u)|^2 dx + C \int_{\delta/3 \leq |x| \leq \delta/2} |x|^{-2m+4-n} |\Delta(\chi u)|^2 dx,
\end{aligned} \tag{3.2}$$

where the constant C' depends on n .

By carefully checking terms on both sides of (3.2), we now choose $\gamma = \sqrt{C'}$ and thus

$$R = \frac{1}{\gamma m} = \frac{1}{\sqrt{C'} m} \quad \text{and} \quad r_4^2 R^2 = \frac{R_0^2 (R_0 + 1)^2}{16 m^2 C'}.$$

Hence, choosing $R_0 < 1$ (suffices to guarantee $R_0^2 (R_0 + 1)^2 / 16 < 1/2$), $m \geq \tilde{m}_0 = \tilde{m}_0(R_0)$, and C_2 sufficiently small such that

$$\frac{1}{\sqrt{C'} m} \leq R_0, \quad \frac{m^2}{2} > C' C_1^2, \quad \text{and} \quad 1 - C' C_2^2 > \frac{1}{2},$$

we can remove the first three terms on the right hand side of the last inequality in (3.2) and obtain

$$\begin{aligned}
&\sum_{|\alpha| \leq 2} m^{2-2|\alpha|} \int_{\delta/2 < |x| < r_4 R} |x|^{-2m+2|\alpha|-n} |D^\alpha u|^2 dx \\
&\leq 2C \int_{\delta/3 < |x| < \delta/2} |x|^{-2m+4-n} |\Delta(\chi u)|^2 dx \\
&\quad + 2C \int_{r_4 R < |x| < 2r_4 R} |x|^{-2m+4-n} |\Delta(\chi u)|^2 dx.
\end{aligned} \tag{3.3}$$

In view of the definition of χ , it is easy to see that for all multiindex α

$$\begin{cases} |D^\alpha \chi| = O(\delta^{-|\alpha|}) \text{ for all } \delta/3 < |x| < \delta/2, \\ |D^\alpha \chi| = O((r_4 R)^{-|\alpha|}) \text{ for all } r_4 R < |x| < 2r_4 R. \end{cases} \tag{3.4}$$

Note that $R_0^2 \leq r_4$ provided $R_0 \leq 1/15$. Therefore, using (3.4) and (2.3) in (3.3), we derive

$$\begin{aligned}
& m^2(2\delta)^{-2m-n} \int_{\delta/2 < |x| \leq 2\delta} |u|^2 dx + m^2(R_0^2 R)^{-2m-n} \int_{2\delta < |x| \leq R_0^2 R} |u|^2 dx \\
& \leq \sum_{|\alpha| \leq 2} m^{2-2|\alpha|} \int_{\delta/2 < |x| < r_4 R} |x|^{-2m+2|\alpha|-n} |D^\alpha u|^2 dx \\
& \leq \tilde{C} \sum_{|\alpha| \leq 2} \delta^{-4+2|\alpha|} \int_{\delta/3 < |x| < \delta/2} |x|^{-2m+4-n} |D^\alpha u|^2 dx \\
& \quad + C'' \sum_{|\alpha| \leq 2} (r_4 R)^{-4+2|\alpha|} \int_{r_4 R < |x| < 2r_4 R} |x|^{-2m+4-n} |D^\alpha u|^2 dx \\
& \leq \tilde{C}' \delta^{-2m-n} \int_{|x| \leq \delta} |u|^2 dx + C'' (r_4 R)^{-2m-n} \int_{|x| \leq R_0 R} |u|^2 dx, \tag{3.5}
\end{aligned}$$

where \tilde{C}' and C'' are independent of R_0 , R , and m .

We then add $m^2(2\delta)^{-2m-n} \int_{|x| \leq \delta/2} |u|^2 dx$ to both sides of (3.5) and obtain

$$\begin{aligned}
& \frac{1}{2} m^2(2\delta)^{-2m-n} \int_{|x| \leq 2\delta} |u|^2 dx + m^2(R_0^2 R)^{-2m-n} \int_{|x| \leq R_0^2 R} |u|^2 dx \\
& = \frac{1}{2} m^2(2\delta)^{-2m-n} \int_{|x| \leq 2\delta} |u|^2 dx + m^2(R_0^2 R)^{-2m-n} \int_{|x| \leq 2\delta} |u|^2 dx \\
& \quad + m^2(R_0^2 R)^{-2m-n} \int_{2\delta < |x| \leq R_0^2 R} |u|^2 dx \\
& \leq \frac{1}{2} m^2(2\delta)^{-2m-n} \int_{|x| \leq 2\delta} |u|^2 dx + \frac{1}{2} m^2(2\delta)^{-2m-n} \int_{|x| \leq 2\delta} |u|^2 dx \\
& \quad + m^2(R_0^2 R)^{-2m-n} \int_{2\delta < |x| \leq R_0^2 R} |u|^2 dx \\
& \leq (\tilde{C}' + m^2) \delta^{-2m-n} \int_{|x| \leq \delta} |u|^2 dx + C'' (r_4 R)^{-2m-n} \int_{|x| \leq R_0 R} |u|^2 dx \\
& = (\tilde{C}' + m^2) \delta^{-2m-n} \int_{|x| \leq \delta} |u|^2 dx \\
& \quad + m^2(R_0^2 R)^{-2m-n} C'' m^{-2} \left(\frac{R_0^2}{r_4}\right)^{2m+n} \int_{|x| \leq R_0 R} |u|^2 dx. \tag{3.6}
\end{aligned}$$

We first observe that

$$\begin{aligned} C'' m^{-2} \left(\frac{R_0^2}{r_4} \right)^{2m+n} &= C'' m^{-2} \left(\frac{4R_0}{R_0 + 1} \right)^{2m+n} \\ &\leq C'' m^{-2} (4R_0)^{2m+n} \leq \exp(-2m) \end{aligned}$$

for all $R_0 \leq 1/16$ and $m^2 \geq C''$. Thus, we obtain that

$$\begin{aligned} &\frac{1}{2} m^2 (2\delta)^{-2m-n} \int_{|x| \leq 2\delta} |u|^2 dx + m^2 (R_0^2 R)^{-2m-n} \int_{|x| \leq R_0^2 R} |u|^2 dx \\ &\leq (\tilde{C}' + m^2) \delta^{-2m-n} \int_{|x| \leq \delta} |u|^2 dx \\ &\quad + m^2 (R_0^2 R)^{-2m-n} \exp(-2m) \int_{|x| \leq R_0 R} |u|^2 dx. \end{aligned} \quad (3.7)$$

It should be noted that (3.7) is valid for all $m = j + \frac{1}{2}$ with $j \in \mathbb{N}$ and $j \geq j_0$, where j_0 depends on R_0 . Setting $R_j = (\gamma(j + \frac{1}{2}))^{-1}$ and using the relation $m = (\gamma R)^{-1}$, we get from (3.7) that

$$\begin{aligned} &\frac{1}{2} m^2 (2\delta)^{-2m-n} \int_{|x| \leq 2\delta} |u|^2 dx + m^2 (R_0^2 R_j)^{-2m-n} \int_{|x| \leq R_0^2 R_j} |u|^2 dx \\ &\leq (\tilde{C}' + m^2) \delta^{-2m-n} \int_{|x| \leq \delta} |u|^2 dx \\ &\quad + m^2 (R_0^2 R_j)^{-2m-n} \exp(-2cR_j^{-1}) \int_{|x| \leq R_0 R_j} |u|^2 dx \end{aligned} \quad (3.8)$$

for all $j \geq j_0$ and $c = \gamma^{-1}$. We now observe that

$$R_{j+1} < R_j < 2R_{j+1} \quad \text{for all } j \in \mathbb{N}.$$

Thus, if $R_{j+1} < R \leq R_j$, we can conclude that

$$\begin{cases} \int_{|x| \leq R_0^2 R} |u|^2 dx \leq \int_{|x| \leq R_0^2 R_j} |u|^2 dx, \\ \exp(-2cR_j^{-1}) \int_{|x| \leq R_0 R_j} |u|^2 dx \leq \exp(-cR^{-1}) \int_{|x| \leq R} |u|^2 dx, \end{cases} \quad (3.9)$$

where we have used the inequality $R_0 R_j \leq 2R_{j+1}/16 < R_{j+1}$ to derive the second inequality above. Namely, we have from (3.8) and (3.9) that

$$\begin{aligned}
& \frac{1}{2} m^2 (2\delta)^{-2m-n} \int_{|x| \leq 2\delta} |u|^2 dx + m^2 (R_0^2 R_j)^{-2m-n} \int_{|x| \leq R_0^2 R} |u|^2 dx \\
& \leq (\tilde{C}' + m^2) \delta^{-2m-n} \int_{|x| \leq \delta} |u|^2 dx \\
& \quad + m^2 (R_0^2 R_j)^{-2m-n} \exp(-cR^{-1}) \int_{|x| \leq R} |u|^2 dx.
\end{aligned} \tag{3.10}$$

If there exists $s \in \mathbb{N}$ such that

$$R_{j+1} < R_0^{2s} \leq R_j \quad \text{for some } j \geq j_0, \tag{3.11}$$

then replacing R by R_0^{2s} in (3.10) leads to

$$\begin{aligned}
& \frac{1}{2} m^2 (2\delta)^{-2m-n} \int_{|x| \leq 2\delta} |u|^2 dx + m^2 (R_0^2 R_j)^{-2m-n} \int_{|x| \leq R_0^{2s+2}} |u|^2 dx \\
& \leq (\tilde{C}' + m^2) \delta^{-2m-n} \int_{|x| \leq \delta} |u|^2 dx \\
& \quad + m^2 (R_0^2 R_j)^{-2m-n} \exp(-cR_0^{-2s}) \int_{|x| \leq R_0^{2s}} |u|^2 dx.
\end{aligned} \tag{3.12}$$

Here s and R_0 are yet to be determined. The trick now is to find suitable s and R_0 satisfying (3.11) and the inequality

$$\exp(-cR_0^{-2s}) \int_{|x| \leq R_0^{2s}} |u|^2 dx \leq \frac{1}{2} \int_{|x| \leq R_0^{2s+2}} |u|^2 dx \tag{3.13}$$

holds with such choices of s and R_0 .

It is time to use the three-sphere inequality (1.2). To this end, we choose $r_1 = R_0^{2k+2}$, $r_2 = R_0^{2k}$ and $r_3 = R_0^{2k-2}$ for $k \geq 1$. Note that $r_1/r_3 < r_2/r_3 \leq R_0^2 \leq 1/4$. Thus (1.2) implies

$$\int_{|x| < R_0^{2k}} |u|^2 dx / \int_{|x| < R_0^{2k+2}} |u|^2 dx \leq C^{1/\tau} \left(\int_{|x| < R_0^{2k-2}} |u|^2 dx / \int_{|x| < R_0^{2k}} |u|^2 dx \right)^a, \tag{3.14}$$

where

$$C = \max\{C_0 R_0^{-2n}, \exp(\beta_0(-1 - 4 \log R_0))\}$$

and

$$\begin{aligned} a = \frac{1 - \tau}{\tau} = \frac{A}{B} &= \frac{(\log(r_1/r_3) - 1)^2 - (\log(r_2/r_3))^2}{-1 - 2 \log(r_2/r_3)} \\ &= \frac{(4 \log R_0 - 1)^2 - (2 \log R_0)^2}{-1 - 4 \log R_0}. \end{aligned}$$

It is not hard to see that

$$\begin{cases} 1 < C \leq C_0 R_0^{-\beta_1}, \\ 2 < a \leq -4 \log R_0, \end{cases} \quad (3.15)$$

where $\beta_1 = \max\{2n, 4\beta_0\}$. Combining (3.15) and using (3.14) recursively, we have that

$$\begin{aligned} & \int_{|x| \leq R_0^{2s}} |u|^2 dx / \int_{|x| \leq R_0^{2s+2}} |u|^2 dx \\ & \leq C^{1/\tau} \left(\int_{|x| < R_0^{2s-2}} |u|^2 dx / \int_{|x| < R_0^{2s}} |u|^2 dx \right)^a \\ & \leq C^{\frac{a^{s-1}-1}{\tau(a-1)}} \left(\int_{|x| < R_0^2} |u|^2 dx / \int_{|x| < R_0^4} |u|^2 dx \right)^{a^{s-1}} \end{aligned} \quad (3.16)$$

for all $s \geq 1$. Now from the definition of a , we have $\tau = 1/(a+1)$ and thus

$$\frac{a^{s-1} - 1}{\tau(a-1)} = \frac{a+1}{a-1} (a^{s-1} - 1) \leq 3a^{s-1}.$$

Then it follows from (3.16) that

$$\begin{aligned} & \int_{|x| \leq R_0^{2s}} |u|^2 dx / \int_{|x| \leq R_0^{2s+2}} |u|^2 dx \\ & \leq C^{3(-4 \log R_0)^{s-1}} \left(\int_{|x| < R_0^2} |u|^2 dx / \int_{|x| < R_0^4} |u|^2 dx \right)^{a^{s-1}} \\ & \leq (C_0^3 (R_0)^{-3\beta_1})^{(-4 \log R_0)^{s-1}} \left(\int_{|x| < R_0^2} |u|^2 dx / \int_{|x| < R_0^4} |u|^2 dx \right)^{a^{s-1}}. \end{aligned} \quad (3.17)$$

Thus, by (3.17), we can get that

$$\begin{aligned}
& \exp(-cR_0^{-2s}) \int_{|x| \leq R_0^{2s}} |u|^2 dx \\
& \leq \exp(-cR_0^{-2s}) (C_0^3(R_0)^{-3\beta_1})^{(-4 \log R_0)^{s-1}} \\
& \quad \left(\int_{|x| < R_0^2} |u|^2 dx / \int_{|x| < R_0^4} |u|^2 dx \right)^{a^{s-1}} \int_{|x| \leq R_0^{2s+2}} |u|^2 dx.
\end{aligned} \tag{3.18}$$

Let $\mu = -\log R_0$, then if R_0 ($\leq 1/16$) is sufficiently small, i.e., μ is sufficiently large, we can see that

$$2t\mu > (t-1) \log(4\mu) + \log(\log C_0^3 + 3\beta_1\mu) - \log(c/4)$$

for all $t \in \mathbb{N}$. In other words, we have that for R_0 small

$$(C_0^3 R_0^{-3\beta_1})^{(-4 \log R_0)^{t-1}} < \exp(cR_0^{-2t}/4) < (1/2) \exp(cR_0^{-2t}/2) \tag{3.19}$$

for all $t \in \mathbb{N}$. We now fix such R_0 so that (3.19) holds. The constants $m_0(R_0)$ and $j_0(R_0)$ are fixed as well. It is a key step in our proof that we can find a universal constant R_0 . After fixing R_0 , we then define a number t_0 , depending on R_0 and u , as

$$\begin{aligned}
t_0 = & \inf\{t \in \mathbb{R} : t \geq (\log 2 - \log(ac) + \log \log(\int_{|x| < R_0^2} |u|^2 dx / \int_{|x| < R_0^4} |u|^2 dx)) \\
& \times (-2 \log R_0 - \log a)^{-1}\}.
\end{aligned}$$

By (3.15), one can easily check that $-2 \log R_0 - \log a > 0$ for all $R_0 \leq 1/16$. With the choice of t_0 , we can see that

$$\left(\int_{|x| < R_0^2} |u|^2 dx / \int_{|x| < R_0^4} |u|^2 dx \right)^{a^{t-1}} \leq \exp(cR_0^{-2t}/2) \tag{3.20}$$

for all $t \geq t_0$.

Let s_1 be the smallest positive integer such that $s_1 \geq t_0$. If

$$R_0^{2s_1} \leq R_{j_0} = (\gamma(j_0 + 1/2))^{-1}, \tag{3.21}$$

then we can find a $j_1 \in \mathbb{N}$ with $j_1 \geq j_0$ such that (3.11) holds, i.e.,

$$R_{j_1+1} < R_0^{2s_1} \leq R_{j_1}.$$

On the other hand, if

$$R_0^{2s_1} > R_{j_0}, \quad (3.22)$$

then we pick the smallest positive integer $s_2 > s_1$ such that $R_0^{2s_2} \leq R_{j_0}$ and thus we can also find a $j_1 \in \mathbb{N}$ with $j_1 \geq j_0$ for which (3.11) holds. We now define

$$s = \begin{cases} s_1 & \text{if (3.21) holds,} \\ s_2 & \text{if (3.22) holds.} \end{cases}$$

It is important to note that with such s , (3.11) is satisfied for some j_1 and (3.19), (3.20) hold. Therefore, we set $m_1 = n + 2(j_1 + 1/2)$ and $m = (m_1 - n)/2$. Combining (3.18), (3.19) and (3.20) yields that

$$\begin{aligned} & \exp(-cR_0^{-2s}) \int_{|x| \leq R_0^{2s}} |u|^2 dx \\ & \leq \exp(-cR_0^{-2s}) (C_0^3(R_0)^{-3\beta_1})^{(-3 \log R_0)^{s-1}} \\ & \quad \left(\int_{|x| < R_0^2} |u|^2 dx / \int_{|x| < R_0^4} |u|^2 dx \right)^{a(s-1)} \int_{|x| \leq R_0^{2s+2}} |u|^2 dx. \\ & \leq \frac{1}{2} \int_{|x| \leq R_0^{2s+2}} |u|^2 dx \end{aligned}$$

which is (3.13). Using (3.13) in (3.12), we have that

$$\begin{aligned} & \frac{1}{2} m^2 (2\delta)^{-2m-n} \int_{|x| \leq 2\delta} |u|^2 dx + \frac{1}{2} m^2 (R_0^2 R_{j_1})^{-2m-n} \int_{|x| \leq R_0^{2s+2}} |u|^2 dx \\ & \leq (\tilde{C}' + m^2) \delta^{-2m-n} \int_{|x| \leq \delta} |u|^2 dx. \end{aligned} \quad (3.23)$$

From (3.23), we get that

$$\frac{(m_1 - n)^2}{8\tilde{C}' + 2(m_1 - n)^2} (R_0^2 R_{j_1})^{-m_1} \int_{|x| \leq R_0^{2s+2}} |u|^2 dx \leq \delta^{-m_1} \int_{|x| \leq \delta} |u|^2 dx \quad (3.24)$$

and

$$\frac{1}{2} m^2 (2\delta)^{-2m-n} \int_{|x| \leq 2\delta} |u|^2 dx \leq (\tilde{C}' + m^2) \delta^{-2m-n} \int_{|x| \leq \delta} |u|^2 dx$$

which implies

$$\int_{|x| \leq 2\delta} |u|^2 dx \leq \frac{8\tilde{C}' + 2(m_1 - n)^2}{(m_1 - n)^2} 2^{m_1} \int_{|x| \leq \delta} |u|^2 dx. \quad (3.25)$$

The estimates (3.24) and (3.25) are valid for all $\delta \leq R_0^{2s+2}/4$. Therefore, (1.3) holds with $R_2 = R_0$. (1.4) holds with $R_3 = R_0^{2s+2}/8$ and $C_3 = \frac{8\tilde{C}'+2(m_1-n)^2}{(m_1-n)^2}2^{m_1}$ and the proof is now complete. \square

References

- [1] G. Alessandrini, E. Beretta, E. Rosset, and S. Vessella, *Optimal stability for elliptic boundary value problems with unknown boundaries*, Ann. Scuola Norm. Sup. Pisa Cl. Sci, **29** (2000), 755-786.
- [2] A. Alinhac, *Non-unicité pour des opérateurs différentiels à caractéristiques complexes simples*, Ann. Sci. E.N.S., **13** (1980), 385-393.
- [3] S. Alinhac and M.S. Baouendi, *A counterexample to strong uniqueness for partial differential equations of Schrödinger's type*. Comm. in PDE **19** (1994), 1727-1733.
- [4] H. Donnelly and C. Fefferman, *Nodal sets of eigenfunctions on Riemannian manifolds*, Invent. Math. **93** (1988), 161-183.
- [5] N. Garofalo and F.H. Lin, *Monotonicity properties of variational integrals, A_p weights and unique continuation*, Indiana Univ. Math. J. **35** (1986), 245-267.
- [6] N. Garofalo and F.H. Lin, *Unique continuation for elliptic operators: a geometric-variational approach*, Comm. Pure Appl. Math., **40**, 347-366, 1987.
- [7] R. Hardt and L. Simon, *Nodal sets for solutions of elliptic equations*, J. Diff. Geom., **30** (1989), 505-522.
- [8] L. Hörmander, "The analysis of linear partial differential operators", Vol. 3, Springer-Verlag, Berlin/New York, 1985.
- [9] D. Jerison and C. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operators. With an appendix by E. M. Stein*, Ann. of Math. (2), **121** (1985), 463-494.
- [10] H. Koch and D. Tataru, *Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients*, Comm. Pure Appl. Math., **54** (2001), 339-360.

- [11] I. Kukavica, *Quantitative uniqueness for second order elliptic operators*, Duke Math. J. **91** (1998), 225-240.
- [12] F.H. Lin, *A uniqueness theorem for parabolic equations*, Comm. Pure Appl. Math., **43** (1990), 127-136.
- [13] F.H. Lin, *Nodal sets of solutions of elliptic and parabolic equations*, Comm. Pure Appl. Math., **44** (1991), 287-308.
- [14] A. Pliš, *On non-uniqueness in Cauchy problem for an elliptic second order differential equation*, Bull. Acad. Pol. Sci., **11** (1963), 95-100.
- [15] R. Regbaoui, *Strong uniqueness for second order differential operators*, J. Diff. Eq. **141** (1997), 201–217.
- [16] T. Wolff, *A counterexample in a unique continuation problem*, Comm. Anal. Geom. **2** (1994), 79-102.